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NEIGHBORLY COMBINATORIAL 3-MANIFOLDS WITH 9 VERTICES

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Abstract. A complete classification is given for neighborly combinatorial 3-manifolds with 9 vertices. It is found that there are 51 types, only one of which is not a sphere.

1. Introduction

A *combinatorial n-sphere* is a finite simplicial n-complex whose body (i.e., the union of its members) is homeomorphic to the topological n-sphere. A *combinatorial n-manifold* is a finite simplicial n-complex M such that for every vertex $v \in M$, the link (v, M) of v in M is a combinatorial $(n-1)$-sphere. Henceforth, all spheres and manifolds to which we refer are combinatorial and connected, and all the complexes are simplicial. A particular case of an n-sphere is the boundary complex of any simplicial $(n+1)$-dimensional convex polytope. A manifold that is isomorphic to such a boundary complex is said to be *polytopal*. A polytopal manifold is necessarily a sphere. But it has been known for a long time (see [11] for references), that not every 3-sphere is polytopal. A non-polytopal 3-sphere with a small number of vertices was first found in [12].

Much work has recently been done on the enumeration of polytopal 2-manifolds (i.e., simplicial 3-polytopes, see [11] for references). Similar combinatorial problems concerning 2-manifolds of more general type have also received a great deal of attention. An example of which is the

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difficult and successful determination (done mainly by Ringel and Youngs [13]) of the number of vertices in a minimal triangulation of a given topological 2-manifold – a problem that was inspired by Heawood's Theorem, the generalization of the famous Four Color Conjecture. For $n > 2$, combinatorial problems concerning n-manifolds have received very little attention. To the authors knowledge even the natural problem of determining the minimal n such that there is a 3-manifold with n vertices that is not a sphere has not been considered. In the present work this number is shown to be 9.

The known information about the enumeration of 3-manifolds with few vertices is presented in Table 1, which summarizes work by Brückner [8], Grünbaum and Sreedharan [12], Grünbaum [10], Barnette [5] Altshuler and Steinberg [4], and Altshuler [3]. The two starred entries in Table 1 are obtained in the present work, whose main objective is their determination. One should also mention the pioneering work in enumerating the simplicial 4-polytopes with up to 8 vertices done by Brückner starting in 1893 (see [10]). His enumeration for the case with 8 vertices (see [8]) has been found to be incorrect and incomplete, and has been corrected by Grünbaum and Sreedharan [12].

A particular (and important) case of a 3-manifold is the *neighborly* 3-manifold. A 3-manifold M is *neighborly* if for every two vertices $x, y \in M$, the edge xy is in M.

The main theorem to be established is:

Theorem 1.1. *There are exactly* 51 *different types of neighborly 3-manifolds with* 9 *vertices, only one of which is not a sphere.*

In view of [3], where it is shown that every 3-manifold with up to 8 vertices is a sphere, Theorem 1.1 implies:

Theorem 1.2. *The minimal n such that there exists a 3-manifold with n vertices that is not a sphere is $n = 9$.*

For every complex C and every simplex $\Delta \in C$, we define star (Δ, C) to be the union of those simplices in C that contain Δ as a face; cl star (Δ, C) is defined to be the smallest complex that contains star (Δ, C) (i.e., the members of cl star (Δ, C) are those of star (Δ, C) and their

Table 1
The known numbers of different types of 3-manifolds having a given number of vertices. The table summarizes works by Brückner (see [10]), Grünbaum - Sreedharan [12], Grünbaum [10], Barnette [5], Altshuler - Steinberg [4] and Altshuler [3]. The values * are determined in the present work.

	Vertices 5	6	7	8	9
Simplicial 4-polytopes	1	2	5	37	
Neighborly 4-polytopes	1	1	1	3	23
3-spheres	1	2	5	39	
Neighborly 3-spheres	1	1	1	4	50*
3-manifolds	1	2	5	39	
Neighborly 3-manifolds	1	1	1	4	51*

faces); antistar (Δ, C) is the complex of those members of C that have no vertex in common with Δ, and the *link* link(Δ, C) of Δ in C is cl star $(\Delta, C) \cap$ antistar (Δ, C). Usually, C will be a manifold and Δ a vertex. The valence of a vertex $v \in C$ is the number of edges (1-simplices) in C incident to v. The difference $C - C^1$ of two complexes C and C^1 is the smallest complex containing the simplices which are in C but not in C^1.

Following Alexander [1], we define a *3-element* to be any 3-complex C whose body is homeomorphic to a topological 3-simplex Δ_3. If ψ is that homeomorphism, then the *boundary* bd C of C is the complex of all the simplices $\Delta \in C$ such that $\psi(\Delta)$ lies on the boundary of Δ_3. (It is clear, however, that bd C is independent of the particular homeomorphism ψ.)

In Section 2, some preliminary results concerning 3-manifolds are discussed. In particular, the concept of a *directly obtainable* 3-manifold is defined and investigated. In Section 3, the methods of [3, 4] are combined to yield the proof of Theorem 1.1, and the reader is frequently asked to consult those two papers for more details. Moreover, Tables 2 and 3 describe only 28 neighborly 3-manifolds with 9 vertices. The description of the remaining 23 manifolds (namely, all the polytopal cases) can be found in [4]. We conclude in Section 4 with some remarks and open questions.

2. Directly obtainable manifolds

A natural approach to the problem of finding all the neighborly 3-manifolds with nine vertices consists of a modification of the method used in [4] to find all the neighborly 4-polytopes with nine vertices from those with eight vertices. For a given 3-manifold M with $n+1$ vertices, we would like to remove one vertex v (together with star(v, M)), and then complete the complex $M -$ star(v, M) to a 3-manifold M' with n vertices. The "hole" created in M by removing star(v, M) should be replaced by some 3-element C such that bd $C =$ link(v, M), all the vertices of C are in bd C, and $M' =$ antistar$(v, M) \cup C$ is a 3-manifold.

If this can be done, we say that M can be *refilled* at the vertex v, and that C is the *refill*. In this case, we also say that M is *directly obtainable at the vertex* v from M' since M can be obtained from M' by replacing the subcomplex C of M' by another complex which has one new vertex v, and which will be cl star(v, M). (The last operation that yields M from M' is the generalized stellar subdivision of M' mentioned in [10].)

Notice that if the 3-manifold M can be refilled at a vertex $v \in M$ and if C is the refill, then $C \cap$ antistar$(v, M) =$ link(v, M); i.e., no interior simplex of C is in antistar(v, M). This follows easily from the fact [1, Theorem 12.5] that if M is an n-manifold or an n-element and Δ is an i-simplex in $M -$ bd M, then link(Δ, M) is an $(n-i-1)$-sphere. (According to our definition, if M is an n-manifold, then bd $M = \emptyset$.)

If M and M' are 3-manifolds with $n+1$ and n vertices, respectively, and there is a vertex in M at which M is directly obtainable from (a 3-manifold isomorphic to) M', we say that M is *directly obtainable from* M'. If M is a 3-manifold and there is some 3-manifold M' from which M is directly obtainable, we say that M is *directly obtainable*.

It has already been mentioned by Grünbaum [10, Remark 5] that not every 3-sphere with more than 4 vertices is directly obtainable at each of its vertices. It is not even clear that every 3-sphere (with more than 4 vertices) is directly obtainable (see [10, Conjecture 5]). In the present work, we find a 3-manifold (N^9_{51} in Table 2) that is not a sphere. Hence it is not directly obtainable since it is shown in [3] that all 3-manifolds with 8 vertices are spheres.

In this section we investigate, for a given 3-manifold M and a given vertex $v \in M$, necessary conditions for M to be directly obtainable at v.

The concept of a stacked 2-sphere, to be defined and studied now, is of crucial importance for this investigation.

Definition 2.1. Let S be a 2-sphere with n vertices, and let G be the graph of S (i.e., the vertices and the edges of G are the vertices and the edges, respectively, of S). S is *stacked* if the vertices of G can be labelled $v_1, \ldots, v_n$, so that if G_i ($1 \leq i \leq n-3$) denotes the subgraph of G spanned by $v_i, v_{i+1}, \ldots, v_n$, then v_i is of valence 3 in G_i ($1 \leq i \leq n-3$).[1]

As an example, it can be easily checked that the 2-spheres $e_1, \ldots, e_7$ presented in Fig. 1 by their Schlegel diagrams are stacked, while $e_8, \ldots, e_{14}$ are not stacked. For a stacked 2-sphere S and an appropriate labelling $v_1, \ldots, v_n$ of its vertices, let Δ_i ($1 \leq i \leq n-3$) denote the 3-simplex whose vertices are v_i and the three vertices of S joined to v_i in G_i. Then $C = \bigcup_{i=1}^{n-3} \Delta_i$ is a 3-element with boundary S, and all the vertices and edges of C are in S. The number of inner faces in C (i.e., 2-simplices in $C - S$) is easily seen to be $n-4$. Moreover, it is clear that a 2-sphere is stacked if and only if it is isomorphic to the boundary complex of a stacked 3-polytope (see [2, Section 1] and [4, Section 2]). It follows that:

(1) The 3-element C associated with a stacked 2-sphere S in the above manner is uniquely defined by S, and does not depend on the labelling of the vertices of S.

(2) If C is a 3-element with all its n vertices on the boundary, then the 2-sphere bd C is stacked if and only if C contains precisely $n-3$ 3-simplices (alternatively; if and only if C contains precisely $n-4$ inner faces).

Let the number of i-simplices in a simplicial complex C be denoted by $f_i(C)$. It is well known (see e.g. [7, Theorem 11]) that for every 3-manifold M,

$$(*) \qquad \sum_{i=0}^{3} (-1)^i f_i(M) = 0$$

[1] The referee has kindly pointed out that a stacked 2-sphere is essentially a dissection of a 3-ball as defined in [6].

Lemma 2.2. *Let C be a 3-element with boundary S. If S contains all the vertices and edges of C, then S is stacked.*

Proof. For each i-simplex Δ and vertex $v \notin \Delta$, let $v \vee \Delta$ denote the $(i+1)$-simplex whose vertices are v and the vertices of Δ. For the 2-sphere S, define $v \vee S$ to be the 3-complex whose 3-simplices are $\{v \vee \Delta: \Delta$ is a 2-simplex in $S\}$, where v is a vertex not in S. Suppose C (and therefore also S) contains n vertices. It follows from Euler's equation for S, that S contains precisely $3n-6$ edges and $2n-4$ 2-simplices. Let x be the number of inner 2-simplices in C.

From [1, Theorem 14.1], it follows that $M = (v \vee S) \cup C$ is a 3-sphere. Since all the edges of C are in S, M contains precisely $n+1$ vertices, $4n-6$ edges and $x+5n-10$ 2-simplices. Hence (*) implies that M contains precisely $x+2n-3$ 3-simplices. Since $2n-4$ of those 3-simplices contain the vertex v, C contains precisely $x+1$ 3-simplices. By a double counting of the 2-simplices in C, we get $4(x+1) = 2n-4+2x$, i.e., $x = n-4$, and it follows from property (2) above that S is stacked.

Theorem 2.3. *Let M be a 3-manifold directly obtainable at a vertex $v \in M$ from a 3-manifold M'. If for every two vertices v_1, v_2 in link (v, M), the manifold M contains the edge $v_1 v_2$, then the 2-sphere link (v, M) is staked.*

Proof. Assume that the condition holds, and let C be the refill, i.e., C is a 3-element such that all the vertices of C are in bd C = link (v, M) and $M' = C \cup$ antistar (v, M). As already mentioned, no inner edge in C is in antistar (v, M). But, since all the vertices of C are in link (v, M), and every edge in C is in M and therefore in antistar (v, M), it follows that all the edges of C are in bd C = link (v, M), and the desired conclusion follows from the lemma.

Theorem 2.4. *Let M be a neighborly 3-manifold directly obtainable at a vertex $v \in M$ from a 3-manifold M'. Then link (v, M) is stacked and M' is neighborly.*

Proof. Since M is neighborly, it follows immediately from Theorem 2.3 that link (v, M) is stacked. If v_1, v_2 are vertices in M', then $v_1 \neq v \neq v_2$

and the edge v_1v_2 is in M. Hence it is also in antistar(v, M) and therefore in M'.

Theorem 2.5. *Let M be a 3-manifold directly obtainable at a vertex $v \in M$ from a 3-manifold M'. Then M is a sphere iff M' is a sphere.*

Proof. Let C be the 3-element used to refill M at v. Cl star(v, M) is a 3-element. If M is a sphere, then by [1, Theorem 14.2] antistar(v, M) is a 3-element, and since it has no internal simplex in common with C, [1, Theorem 14.1] implies that antistar$(v, M) \cup C = M'$ is a sphere. Conversely, if M' is a sphere, then by [1, Theorem 14.2], antistar(v, M) is a sphere, and by [1, Theorem 14.1], antistar$(v, M) \cup$ star$(v, M) = M$ is a sphere.

Remark 2.6. The manifolds M, M' of Theorem 2.5 can easily be seen to be of the same homotopy type, but this result is not needed here.

Corollary 2.7. A 3-manifold with nine vertices that is not a sphere is not directly obtainable.

Proof. This is an immediate consequence of Theorem 2.5 since the main theorem in [3] states that every 3-manifold with eight vertices is a sphere.

In the case in which the manifold M of Theorem 2.4 is polytopal, i.e., is isomorphic to the boundary complex of some convex 4-polytope P (so that M is a sphere), and v is any vertex of M, the intrinsic structure of the Euclidean 4-dimensional space $\mathbf{R}^4$ implies that M is directly obtainable at v. This is done by placing the vertices of P in general position in $\mathbf{R}^4$ (i.e., no four vertices are in one plane). If P^* is the convex hull of the vertices of P other than V (where V is the vertex of P that corresponds to v), then the boundary complex of the complex of all proper faces of P^* seen from V is isomorphic to both link(v, M) and to the boundary complex of the vertex figure of P at V, and M is directly obtainable (at v) from bd P^* (see [10, Remark 5]). Thus [4, Theorem 1] follows from our Theorem 2.4.

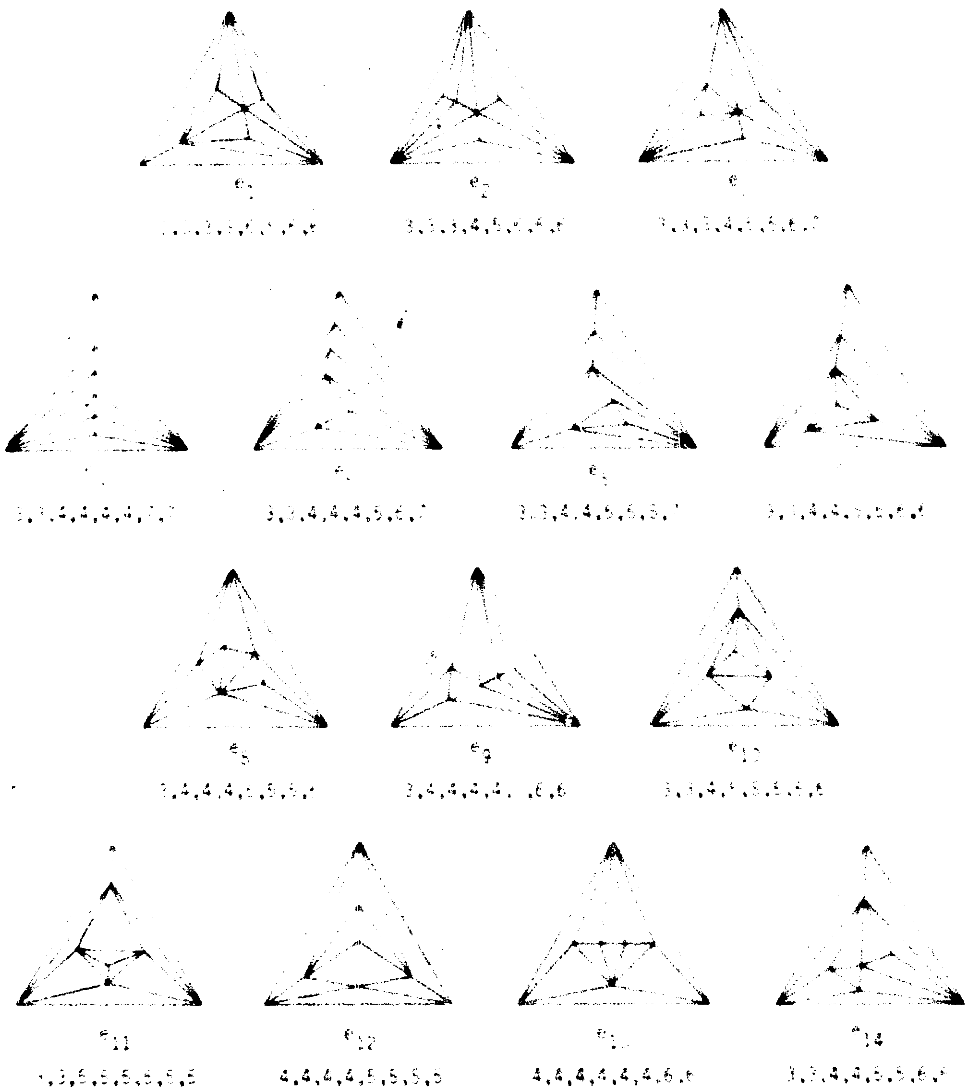

Fig. 1. The 14 2-spheres with 8 vertices and the valences of their vertices.

3. Proof of Theorem 1.1.

First we find the neighborly 3-manifolds with nine vertices (briefly referred to as the N^9's) that are directly obtainable. By Corollary 2.7., all those N^9's are spheres, and by Theorem 2.4 they are directly obtainable from neighborly 3-spheres with 8 vertices. The list of all 39 3-spheres with eight vertices was partially obtained by Grünbaum and Sreedharan [12] and completed by Barnette [5]. This list contains precisely four neighborly 3-spheres with eight vertices, namely P^8_{35}, P^8_{36}, P^8_{37} and M of [12].

Let $\mathcal{N}^9$ be directly obtainable at a vertex $v \in \mathcal{N}^9$ from a 3-manifold $\mathcal{N}^8$. Then $\mathcal{N}^8$ is one of the four 3-spheres mentioned above, and, by Theorem 2.4, S = link (v, $\mathcal{N}^9$) is a stacked 2-sphere which has eight vertices because $\mathcal{N}^9$ is neighborly. The 14 possible 2-spheres with eight vertices are shown in Fig. 1. Only the first seven of them (i.e., e_1, ..., e_7) are stacked. Hence S is isomorphic to one of these seven.

Let a pair ($\mathcal{N}^8$, C) be given, where $\mathcal{N}^8$ is a neighborly 3-sphere with eight vertices, and $C \subset \mathcal{N}^8$ is a 3-element with five 3-simplices and eight vertices which lie in bd C. Then bd C is stacked (see (2) in Section 2) and is therefore isomorphic to one of e_1, ..., e_7. The pair ($\mathcal{N}^8$, C) then yields a unique 3-sphere $\mathcal{N}^9 = (\mathcal{N}^8 - C) \cup (v \vee \mathrm{bd}\, C)$, where v is a vertex not in $\mathcal{N}^8$. Therefore, to find all directly obtainable $\mathcal{N}^9$'s, we can find all pairs ($\mathcal{N}^8$, C), classify the resulting $\mathcal{N}^9$'s into isomorphism classes and select one representative from each class.

Notice that this is precisely the procedure used in [4] to find all neighborly 4-polytopes with nine vertices. The only difference is that in [4] there were three possibilities for $\mathcal{N}^8$, while here there is also the case $\mathcal{N}^8 = M$. Unlike [4], where one had to check that each resulting $\mathcal{N}^9$ is isomorphic to the boundary complex of some 4-polytope, no further check is needed here. Thus, the computer program to find the directly obtainable $\mathcal{N}^9$'s is a simple modification of that described in detail in [4, Section 3], and no further description is given here. This program yielded 50 directly obtainable $\mathcal{N}^9$'s. They are denoted by $\mathcal{N}^9_i$ ($1 \le i \le 50$). The first 23 cases $\mathcal{N}^9_1$, ..., $\mathcal{N}^9_{23}$ (which are all the polytopal cases) have already been described in [4, Table 2]. The remaining 27 cases $\mathcal{N}^9_{24}$, ..., $\mathcal{N}^9_{50}$ (as well as $\mathcal{N}^9_{51}$, which is found later) are described in this paper in Table 2.

The vertices of each of the $\mathcal{N}^9$'s are denoted by 1, 2, ..., 9. The description of each of these manifolds consists of the list of 3-simplices it contains; a list for each of the nine vertices of the 3-simplices that contain it (in [4], where all the manifolds are polytopal, those are just the facets of the dual polytope); the types of the links of the vertices (in [4, Table 2], these links are identified by a list of the valences of the vertices in the link, the list C_i corresponds to our 2-sphere e_i ($1 \le i \le 7$)); and finally the *edge-valence matrix* and its determinant. The last two

Table 2
The non-polytopal neighborly 3-manifolds with 9 vertices

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1235	J 1679	S 2679	1 ABCDEFGHIJKL	E11	*	5	5	5	3	5	5	5	3
	B 1236	K 1689	T 3458	2 ABCDEMNOPQRS	E 5	5	*	4	4	6	7	4	3	3
I = 24	C 1245	L 1789	U 3478	3 ABFGHMNTUVWX	E 5	5	4	*	4	6	3	7	4	3
	D 1248	M 2357	V 3579	4 CDFGIOPTUYZ$	E 5	5	4	4	*	6	4	3	7	3
	E 1268	N 2367	W 3589	5 ACFMOQRTVWYZ	E 1	3	6	6	6	*	3	3	3	6
DETERM.=	F 1345	O 2456	X 3789	6 BEHJKNOPQSY$	E 5	5	7	3	4	3	*	4	4	6
64047024	G 1347	P 2468	Y 4569	7 GHIJLMNRSUVX	E 5	5	4	7	3	3	4	*	4	6
	H 1367	Q 2569	Z 4589	8 DEIKLPTUWXZ$	E 5	5	3	4	7	3	4	4	*	6
	I 1478	R 2579	$ 4689	9 JKLQRSVWXYZ$	E 1	3	3	3	3	6	6	6	6	*
	A 1235	J 1579	S 2569	1 ABCDEFGHIJKL	E 3	*	3	5	5	6	3	4	7	3
	B 1238	K 1589	T 2589	2 ABCMNOPQRSTU	E 3	3	*	6	3	7	5	4	5	3
I = 25	C 1258	L 1789	U 2689	3 ABDEFMNOPVWX	E 2	5	6	*	6	3	6	4	3	3
	D 1345	M 2345	V 3469	4 DEGHIMNQVWYZ	E 9	5	3	6	*	4	4	6	4	4
	E 1346	N 2347	W 3479	5 ACDGJKMQRST$	E 3	6	7	3	4	*	3	5	3	5
DETERM.=	F 1368	O 2367	X 3679	6 EFHOPRSUVXY$	E 7	3	5	6	4	3	*	4	5	6
69120000	G 1457	P 2368	Y 4689	7 GIJLNOQRWXZ$	E 9	4	4	4	6	5	4	*	3	6
	H 1468	Q 2457	Z 4789	8 BCFHIKLPTUYZ	E 3	7	5	3	4	3	5	3	*	6
	I 1478	R 2567	$ 5679	9 JKLSTUVWXYZ$	E 2	3	3	3	4	5	6	6	6	*
	A 1245	J 1569	S 2689	1 ABCDEFGHIJKL	E 5	*	4	4	5	7	6	3	4	3
	B 1246	K 1589	T 2789	2 ABCDMNOPQRST	E 5	4	*	4	5	3	6	4	7	3
I = 26	C 1258	L 1689	U 3479	3 EFGHMNOPUVWX	E 5	4	4	*	6	3	5	7	4	3
	D 1268	M 2346	V 3489	4 ABEFIMNQUVYZ	E 8	5	5	6	*	5	3	4	4	4
	E 1346	N 2348	W 3567	5 ACGHIJKQWYZ$	E 6	7	3	3	5	*	4	5	4	5
DETERM.=	F 1347	O 2367	X 3789	6 BDEGJLMORSW$	E14	6	6	5	3	4	*	4	3	5
76266288	G 1356	P 2378	Y 4579	7 FHIOPRTUWXY$	E 5	3	4	7	4	5	4	*	3	6
	H 1357	Q 2458	Z 4589	8 CDKLNPQSTVXZ	E 4	4	7	4	4	4	3	3	*	7
	I 1457	R 2679	$ 5679	9 JKLRSTUVXYZ$	E 3	3	3	3	4	5	5	6	7	*

* Represents the number 12.

Table 2 (continued)

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1234	J 1579	S 2589	1 ABCDEFGHIJKL	E 5	*	4	6	4	7	3	4	5	3
	B 1238	K 1589	T 2689	2 ABCDMNOPQRST	E 5	4	*	4	6	5	4	3	7	3
I = 27	C 1245	L 1789	U 3469	3 ABEFGHMNUVWX	E 3	6	4	*	5	3	5	7	3	3
	D 1258	M 2347	V 3479	4 ACEIMOPQUVYZ	E 9	4	6	5	*	3	6	4	4	4
	E 1346	N 2378	W 3567	5 CDFGIJKORSW$	E 3	7	5	3	3	*	6	4	3	5
DETERM.=	F 1356	O 2456	X 3679	6 EFIOPRTUWXY$	E 2	3	4	5	6	6	*	3	3	6
80478144	G 1357	P 2468	Y 4689	7 GHJLMNQVWXZ$	E 5	4	3	7	4	4	3	*	5	6
	H 1378	Q 2478	Z 4789	8 BDHKLNPQSTYZ	E 3	5	7	3	4	3	3	5	*	6
	I 1456	R 2569	$ 5679	9 JKLRSTUVXYZ$	E 2	3	3	3	4	5	6	6	6	*
	A 1234	J 1569	S 2789	1 ABCDEFGHIJKL	E 7	*	5	6	3	4	5	4	6	3
	B 1236	K 1589	T 3456	2 ABCDEMNOPQRS	E 3	5	*	3	5	3	7	6	4	3
I = 28	C 1247	L 1689	U 3458	3 ABFGHIMTUVWX	E 5	6	3	*	7	4	4	4	5	3
	D 1268	M 2346	V 3479	4 ACFMNOTUVWYZ	E 3	3	5	7	*	6	3	5	3	4
	E 1278	N 2456	W 3489	5 GHJKNOPTUYZ$	E 9	4	3	4	6	*	6	4	4	5
DETERM.=	F 1347	O 2457	X 3789	6 BDGJLMNPQRT$	E 3	5	7	4	3	6	*	3	3	5
84262464	G 1356	P 2567	Y 4579	7 CEFIOPQSVXY$	E 9	4	6	4	5	4	3	*	4	6
	H 1358	Q 2679	Z 4589	8 DEHIKLRSUWXZ	E 5	6	4	5	3	4	3	4	*	7
	I 1378	R 2689	$ 5679	9 JKLQRSVWXYZ$	E 3	3	3	3	4	5	5	6	7	*
	A 1246	J 1569	S 2689	1 ABCDEFGHIJKL	E 9	*	4	5	4	4	6	4	6	3
	B 1247	K 1589	T 2789	2 ABCDMNOPQRST	E 5	4	*	3	5	4	7	6	4	3
I = 29	C 1268	L 1689	U 3458	3 EFGHIMNOUVWX	E 6	5	3	*	7	5	4	4	5	3
	D 1278	M 2345	V 3479	4 ABEFMNPUVWYZ	E 6	4	5	7	*	5	3	5	3	4
	E 1346	N 2346	W 3489	5 GHJKMOPQUYZ$	E 12	4	4	5	5	*	5	4	4	5
DETERM.=	F 1347	O 2356	X 3789	6 ACEGJLNOQRS$	E 3	6	7	4	3	5	*	3	3	5
96526080	G 1356	P 2457	Y 4579	7 BDFIPQRTVXY$	E 9	4	6	4	5	4	3	*	4	6
	H 1358	Q 2567	Z 4589	8 CDHIKLSTUWXZ	E 5	6	4	5	3	4	3	4	*	7
	I 1378	R 2679	$ 5679	9 JKLRSTVWXYZ$	E 3	3	3	3	4	5	5	6	7	*

* Represents the number 12.

Table 2 (continued)

Case N_I^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1234	J 1579	S 2689	1 ABCDEFGHIJKL	E 2	*	5	6	3	6	3	6	4	3
	B 1236	K 1589	T 3458	2 ABCDEMNOPQRS	E 3	5	*	3	5	6	7	3	4	3
I = 30	C 1245	L 1789	U 3469	3 ABFGHIMTUVWX	E 3	6	3	*	7	3	5	5	4	3
	D 1257	M 2346	V 3478	4 ACFMNOTUVWYZ	E 5	3	5	7	*	4	4	3	6	4
	E 1267	N 2458	W 3479	5 CDFGJKNPQRT$	E 7	6	6	3	4	*	3	4	5	5
DETERM.=	F 1345	O 2468	X 3679	6 BEHMOPQSUXY$	E 3	3	7	5	4	3	*	5	3	6
115560000	G 1358	P 2567	Y 4689	7 DEHIJLPVWXZ$	E 7	6	3	5	3	4	5	*	4	6
	H 1367	Q 2569	Z 4789	8 GIKLNORSTVYZ	E 9	4	4	4	6	5	3	4	*	6
	I 1378	R 2589	$ 5679	9 JKLQRSUWXYZ$	E 2	3	3	3	4	5	6	6	6	*
	A 1235	J 1569	S 2789	1 ABCDEFGHIJKL	E 6	*	5	5	3	5	7	4	4	3
	B 1237	K 1589	T 3456	2 ABCDEMNOPQRS	E 6	5	*	5	3	4	4	5	7	3
I = 31	C 1258	L 1689	U 3479	3 ABFGHMNOTUVW	E 6	5	5	*	7	4	3	5	4	3
	D 1267	M 2345	V 3489	4 FGIMNPTUVXYZ	E 5	3	3	7	*	6	4	5	4	4
	E 1268	N 2348	W 3789	5 ACHJKMPTXYZ$	E 8	5	4	4	6	*	5	3	4	5
DETERM.=	F 1346	O 2378	X 4567	6 DEFHIJLQRTX$	E 6	7	4	3	4	5	*	5	3	5
120735936	G 1347	P 2458	Y 4579	7 BDGIOQSUWXY$	E 10	4	5	5	5	3	5	*	3	6
	H 1356	Q 2679	Z 4589	8 CEKLNOPRSVWZ	E 4	4	7	4	4	4	3	3	*	7
	I 1467	R 2689	$ 5679	9 JKLQRSUVWYZ$	E 3	3	3	3	4	5	5	6	7	*
	A 1236	J 1578	S 2468	1 ABCDEFGHIJKL	E 10	*	4	5	3	6	5	5	5	3
	B 1237	K 1589	T 2789	2 ABCDMNOPQRST	E 5	4	*	7	4	3	5	4	6	3
I = 32	C 1268	L 1689	U 3479	3 ABEFGMNOPQUV	E 5	5	7	*	6	4	3	4	3	4
	D 1278	M 2345	V 3489	4 EFHMNRSUVWXY	E 8	3	4	6	*	5	5	5	4	4
	E 1345	N 2348	W 4567	5 EGHIJKMORWZ$	E 7	6	3	4	5	*	6	5	3	4
DETERM.=	F 1347	O 2356	X 4679	6 ACGILORSWXYZ	E 10	5	5	3	5	6	*	3	4	5
122540544	G 1356	P 2379	Y 4689	7 BDFHJPTUWXZ$	E 8	5	4	4	5	5	3	*	4	6
	H 1457	Q 2389	Z 5679	8 CDJKLNQSTVY$	E 5	5	6	3	4	3	4	4	*	7
	I 1569	R 2456	$ 5769	9 IKLPQTUVXYZ$	E 5	3	3	4	4	4	5	6	7	*

* Represents the number 12.

Table 2 (continued)

Case N_I^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1236	J 1579	S 2689	1 ABCDEFGHIJKL	L 11	*	5	5	5	5	3	5	5	3
	B 1237	K 1589	T 3469	2 ABCDEMNOPQRS	L 5	5	*	4	4	7	6	3	4	3
I = 33	C 1245	L 1789	U 3478	3 ABFGHMNTUVWX	L 3	5	4	*	5	3	6	7	3	3
	D 1246	M 2356	V 3479	4 CDFGIOPTUVYZ	L 6	5	4	5	*	3	5	3	7	4
	E 1257	N 2357	W 3567	5 CEIJKMNOQRW$	L 6	5	7	3	3	*	4	5	4	5
DETERM.=	F 1346	O 2458	X 3679	6 ADEMPQSTWXY$	L 2	3	6	6	5	4	*	3	3	6
122652144	G 1348	P 2468	Y 4689	7 BEHJLNUVWXZ$	L 3	5	3	7	3	5	3	*	4	6
	H 1378	Q 2569	Z 4789	8 GHIKLOPRSUYZ	L 5	5	4	3	7	4	3	4	*	6
	I 1458	R 2589	$ 5679	9 JKLQRSTVXYZ$	L 2	3	3	3	4	5	6	6	6	*
	A 1245	J 1579	S 2589	1 ABCDEFGHIJKL	L 9	*	4	5	4	6	4	6	4	3
	B 1246	K 1589	T 2689	2 ABCDMNOPQRST	L 3	4	*	3	5	7	6	3	5	3
I = 34	C 1257	L 1789	U 3469	3 EFGHIMNOUVWX	L 6	5	3	*	7	4	4	5	5	3
	D 1267	M 2345	V 3478	4 ABEFMNPUVWYZ	L 6	4	5	7	*	3	5	3	5	4
	E 1345	N 2348	W 3479	5 ACEGJKMOQRS$	L 5	6	7	4	3	*	3	4	4	5
DETERM.=	F 1346	O 2358	X 3679	6 BDFHPQRTUXY$	L 14	4	6	4	5	3	*	5	3	6
125869824	G 1358	P 2468	Y 4689	7 CDHIJLQVWXZ$	L 7	6	3	5	3	4	5	*	4	6
	H 1367	Q 2567	Z 4789	8 GIKLNOPSTVYZ	L 8	4	5	5	5	4	3	4	*	6
	I 1378	R 2569	$ 5679	9 JKLRSTUWXYZ$	L 2	3	3	3	4	5	6	6	6	*
	A 1234	J 1579	S 2689	1 ABCDEFGHIJKL	L 6	*	5	4	5	5	3	7	4	3
	B 1236	K 1589	T 3458	2 ABCDEMNOPQRS	L 3	5	*	5	3	7	6	3	4	3
I = 35	C 1245	L 1789	U 3468	3 ABFGMNOTUVWX	L 5	4	5	*	7	3	6	4	4	3
	D 1257	M 2345	V 3469	4 ACFHIMTUVWYZ	L 5	5	3	7	*	4	3	4	6	4
	E 1267	N 2358	W 3479	5 CDHJKMNPQRT$	L 6	5	7	3	4	*	3	4	5	5
DETERM.=	F 1347	O 2368	X 3679	6 BEGOPQSUVXY$	L 2	3	6	6	3	3	*	5	4	6
129265344	G 1367	P 2567	Y 4689	7 DEFGIJLPWXZ$	L 5	7	3	4	4	4	5	*	3	6
	H 1458	Q 2569	Z 4789	8 HIKLNORSTUYZ	L 9	4	4	4	6	5	4	3	*	6
	I 1478	R 2589	$ 5679	9 JKLQRSVWXYZ$	L 2	3	3	3	4	5	6	6	6	*

* Represents the number 12.

Table 2 (continued)

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1234	J 1569	S 2689	1 ABCDEFGHIJKL	E 5	*	4	4	6	7	5	3	4	3
	B 1235	K 1589	T 2789	2 ABCDMNOPQRST	E 5	4	*	6	4	3	7	4	5	3
I = 36	C 1246	L 1689	U 3479	3 ABEFMNOPUVWX	E 5	4	6	*	5	4	3	7	4	3
	D 1256	M 2348	V 3489	4 ACEGHIMQUVYZ	E 9	6	4	5	*	4	3	4	6	4
	E 1347	N 2356	W 3567	5 BDFGHJKNWYZ$	E 6	7	3	4	4	*	5	5	3	5
DETERM.=	F 1357	O 2367	X 3789	6 CDIJLNOQRSW$	E 6	5	7	3	3	5	*	4	4	5
132404544	G 1457	P 2378	Y 4579	7 EFGOPRTUWXY$	E 5	3	4	7	4	5	4	*	3	6
	H 1458	Q 2468	Z 4589	8 HIKLMPQSTVXZ	E 5	4	5	4	6	3	4	3	*	7
	I 1468	R 2679	$ 5679	9 JKLRSTUVXYZ$	E 3	3	3	3	4	5	5	6	7	*
	A 1236	J 1579	S 2589	1 ABCDEFGHIJKL	E 7	*	4	5	6	5	3	4	6	3
	B 1238	K 1589	T 2689	2 ABCDMNOPQRST	E 3	4	*	5	3	5	7	3	6	3
I = 37	C 1246	L 1789	U 3457	3 ABEFGMNOUVWX	E 11	5	5	*	5	5	5	5	3	3
	D 1248	M 2357	V 3469	4 CDEFHIPUVWYZ	E 10	6	3	5	*	3	5	5	5	4
	E 1345	N 2358	W 3479	5 EGHIJKMNQRSU$	E 10	5	5	5	3	*	3	6	4	5
DETERM.=	F 1346	O 2367	X 3679	6 ACFOPQRTVXY$	E 3	3	7	5	5	3	*	4	3	6
132698736	G 1358	P 2466	Y 4689	7 HIJLMOQUWXZ$	E 7	4	3	5	5	6	4	*	3	6
	H 1457	Q 2567	Z 4789	8 BDGIKLNPSTYZ	E 2	6	6	3	5	4	3	3	*	6
	I 1478	R 2569	$ 5679	9 JKLRSTVWXYZ$	E 2	3	3	3	4	5	6	6	6	*
	A 1234	J 1579	S 2689	1 ABCDEFGHIJKL	E 7	*	5	6	3	5	4	6	4	3
	B 1235	K 1589	T 3458	2 ABCDEMNOPQRS	E 3	5	*	3	5	7	6	3	4	3
I = 38	C 1246	L 1789	U 3469	3 ABFGHIMTUVWX	E 5	6	3	*	7	4	4	5	4	3
	D 1257	M 2345	V 3478	4 ACFMNOTUVWYZ	E 3	3	5	7	*	3	5	3	6	4
	E 1267	N 2458	W 3479	5 BDGJKMNPQRT$	E 6	5	7	4	3	*	3	4	5	5
DETERM.=	F 1346	O 2468	X 3679	6 CEFHOPQSUXY$	E 14	4	6	4	5	3	*	5	3	6
134175744	G 1358	P 2567	Y 4689	7 DEHIJLPVWXZ$	E 7	6	3	5	3	4	5	*	4	6
	H 1367	Q 2569	Z 4789	8 GIKLNORSTVYZ	E 9	4	4	4	6	5	3	4	*	6
	I 1378	R 2589	$ 5679	9 JKLQRSUWXYZ$	E 2	3	3	3	4	5	6	6	6	*

* Represents the number 12.

Table 2 (continued)

Case N_I^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A 1245	J 1579	S 2569	1 ABCDEFGHIJKL	E 2	*	3	4	6	6	3	5	6	3
	B 1247	K 1589	T 2589	2 ABCMNOPQRSTU	E 6	3	*	5	4	7	5	5	4	3
I = 39	C 1257	L 1789	U 2689	3 DEFGMNOPQVWX	E 9	4	5	*	6	4	6	4	4	3
	D 1345	M 2345	V 3469	4 ABDEHIMNVWYZ	E 9	6	4	6	*	3	4	5	4	4
	E 1346	N 2347	W 3479	5 ACDEJKMORST$	E 5	6	7	4	3	*	3	4	4	5
DETERM.=	F 1358	O 2358	X 3679	6 EGHPQRSUVXY$	E 7	3	5	6	4	3	*	4	5	6
134815680	G 1368	P 2367	Y 4689	7 BCIJLNPRWXZ$	E 8	5	5	4	5	4	4	*	3	6
	H 1468	Q 2368	Z 4789	8 FGHIKLOQTUYZ	E 9	6	4	4	4	4	5	3	*	6
	I 1478	R 2567	$ 5679	9 JKLSTUVWXYZ$	E 2	3	3	3	4	5	6	6	6	*
	A 1234	J 1569	S 2689	1 ABCDEFGHIJKL	E 5	*	4	4	6	7	5	3	4	3
	B 1235	K 1589	T 2789	2 ABCDMNOPQRST	E 5	4	*	6	3	4	5	7	4	3
I = 40	C 1247	L 1689	U 3468	3 ABFGMNOTUVWX	E 7	4	6	*	6	3	5	4	5	3
	D 1257	M 2347	V 3479	4 ACEGHIMUVWYZ	E14	6	3	6	*	4	3	5	5	4
	E 1346	N 2356	W 3489	5 BDFGHJKNQYZ$	E 6	7	4	3	4	*	5	5	3	5
DETERM.=	F 1356	O 2368	X 3789	6 EFIJLNOQRSU$	E11	5	5	5	3	5	*	3	5	5
136538496	G 1457	P 2378	Y 4579	7 CDGMPQRTVXY$	E 3	3	7	4	5	5	3	*	3	6
	H 1458	Q 2567	Z 4589	8 HIKLOPSTUWXZ	E 6	4	4	5	5	3	5	3	*	7
	I 1468	R 2679	$ 5679	9 JKLRSTVWXYZ$	E 3	3	3	3	4	5	5	6	7	*
	A 1234	J 1579	S 2589	1 ABCDEFGHIJKL	E 5	*	4	4	6	5	3	4	7	3
	B 1238	K 1589	T 2689	2 ABCDMNOPQRST	E 3	4	*	6	3	5	7	3	5	3
I = 41	C 1246	L 1789	U 3457	3 ABEFMNOPUVWX	E 7	4	6	*	6	5	4	5	3	3
	D 1268	M 2346	V 3469	4 ACEGHIMUVWYZ	E14	6	3	6	*	3	5	5	4	4
	E 1345	N 2357	W 3479	5 EFGJKNOQRSU$	E10	5	5	5	3	*	3	6	4	5
DETERM.=	F 1358	O 2358	X 3679	6 CDHMPQRTVXY$	E 5	3	7	4	5	3	*	4	4	6
137337264	G 1457	P 2367	Y 4689	7 GIJLNPQUWXZ$	E 7	4	3	5	5	6	4	*	3	6
	H 1468	Q 2567	Z 4789	8 BDFHIKLOSTYZ	E 5	7	5	3	4	4	4	3	*	6
	I 1478	R 2569	$ 5679	9 JKLRSTVWXYZ$	E 2	3	3	3	4	5	6	6	6	*

* Represents the number 12.

Table 2 (continued)

Case A_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A - 1236	J - 1579	S - 2689	1 - ABCDEFGHIJKL	E 7	•	5	6	4	6	3	5	4	3
	B - 1237	K - 1589	T - 3458	2 - ABCDEMNOPQRS	E 5	5	•	3	4	6	7	4	4	3
I = 42	C - 1245	L - 1789	U - 3469	3 - ABFGHIMTUVWX	E 2	6	3	•	6	3	5	6	4	3
	D - 1246	M - 2367	V - 3478	4 - CDFGNOTUVWYZ	F 9	4	4	6	•	4	5	3	6	4
	E - 1257	N - 2458	W - 3479	5 - CEFHJKNPQRT$	E 7	6	6	3	4	•	3	4	5	5
DETERM.=	F - 1345	O - 2468	X - 3679	6 - ADGMOPQSUXY$	E 3	3	7	5	5	3	•	4	3	6
140863104	G - 1346	P - 2567	Y - 4689	7 - BEIJLMPVWXZ$	E 9	5	4	6	3	4	4	•	4	6
	H - 1358	Q - 2569	Z - 4789	8 - HIKLNORSTVYZ	F 9	4	4	4	6	5	3	4	•	6
	I - 1378	R - 2589	$ - 5679	9 - JKLQRSUWXYZ$	E 2	3	3	3	4	5	6	6	6	•
	A - 1234	J - 1569	S - 3456	1 - ABCDEFGHIJKL	E 7	•	6	5	3	5	4	4	6	3
	B - 1235	K - 1589	T - 3458	2 - ABCDEFMNOPQR	E 5	6	•	4	4	3	7	5	4	3
I = 43	C - 1247	L - 1689	U - 3479	3 - ABGHIMNSTUVW	E 6	5	4	•	7	5	3	4	5	3
	D - 1256	M - 2346	V - 3489	4 - ACGMOSTUVXYZ	E 5	3	4	7	•	5	4	6	3	4
	E - 1268	N - 2356	W - 3789	5 - BDHJKNSTXYZ$	E 6	5	3	5	5	•	6	3	4	5
DETERM.=	F - 1278	O - 2467	X - 4567	6 - DEJLMNOPQSX$	E 5	4	7	3	4	6	•	4	3	5
143288064	G - 1347	P - 2679	Y - 4579	7 - CFGIOPRUWXY$	E 9	4	5	4	6	3	4	•	4	6
	H - 1358	Q - 2689	Z - 4589	8 - EFHIKLQRTVWZ	E 5	6	4	5	3	4	3	4	•	7
	I - 1378	R - 2789	$ - 5679	9 - JKLPQRUVWYZ$	E 3	3	3	3	4	5	5	6	7	•
	A - 1234	J - 1569	S - 2578	1 - ABCDEFGHIJKL	E 5	•	4	4	6	5	7	3	4	3
	B - 1236	K - 1589	T - 2789	2 - ABCDMNOPQRST	E 5	4	•	7	3	6	4	5	4	3
I = 44	C - 1245	L - 1689	U - 3458	3 - ABEFMNOPQUVW	E 5	4	7	•	6	3	3	5	4	4
	D - 1256	M - 2345	V - 3479	4 - ACEGHIMUVWXY	E 9	6	3	6	•	4	4	4	5	4
	E - 1347	N - 2358	W - 3489	5 - CDGJKMNRSUZ$	F 9	5	6	3	4	•	4	4	6	4
DETERM.=	F - 1367	O - 2367	X - 4679	6 - BDFHIJLORXYZ	E 5	7	4	3	4	4	•	6	3	5
144061632	G - 1458	P - 2379	Y - 4689	7 - EFHOPRSTVXZ$	F 7	3	5	5	4	4	6	•	3	6
	H - 1467	Q - 2389	Z - 5679	8 - GIKLNQSTUWY$	E 5	4	4	4	5	6	3	3	•	7
	I - 1468	R - 2567	$ - 5789	9 - JKLPQTVWXYZ$	F 5	3	3	4	4	4	5	6	7	•

• Represents the number 12.

Table 2 (continued)

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix
	A 1235	J 1578	S 2789	1 ABCDEFGHIJKL	E 5	* 5 4 3 7 4 4 6 3
	B 1238	K 1589	T 3468	2 ABCDEMNOPQRS	E 6	5 * 7 5 4 3 5 4 3
I = 45	C 1245	L 1689	U 3479	3 ABFGMNOPQTUV	E 6	4 7 * 5 3 5 3 5 4
	D 1247	M 2346	V 3489	4 CDHMNRTUVWXY	E 7	3 5 5 * 4 6 6 3 4
	E 1278	N 2347	W 4567	5 ACFHIJKORWZ$	E 5	7 4 3 4 * 6 5 3 4
DETERM =	F 1356	O 2356	X 4679	6 FGILMORTWXYZ	E14	4 3 5 6 6 * 3 4 5
146109744	G 1368	P 2379	Y 4689	7 DEHJNPSUWXZ$	E14	4 5 3 6 5 3 * 4 6
	H 1457	Q 2389	Z 5679	8 BEGJKLQSTVY$	E 5	6 4 5 3 3 4 4 * 7
	I 1569	R 2456	$ 5789	9 IKLPQSUVXYZ$	E 5	3 3 4 4 4 5 6 7 *
	A 1235	J 1569	S 2689	1 ABCDEFGHIJKL	E 7	* 4 5 6 6 5 3 4 3
	B 1237	K 1589	T 2789	2 ABCDMNOPQRST	E 6	4 * 5 3 5 5 7 4 3
I = 46	C 1245	L 1689	U 3468	3 ABEFGMNOUVWX	E11	5 5 * 5 3 5 5 5 3
	D 1247	M 2356	V 3479	4 CDEFHIPUVWYZ	E10	6 3 5 * 5 3 5 5 4
	E 1346	N 2368	W 3489	5 ACGHJKMPQYZ$	E10	6 5 3 5 * 5 4 3 5
DETERM =	F 1347	O 2378	X 3789	6 FGIJLMNQRSU$	E11	5 5 5 3 5 * 3 5 5
154533883	G 1356	P 2457	Y 4579	7 BDFOPQRTVXY$	E 3	3 7 5 5 4 3 * 3 6
	H 1458	Q 2567	Z 4589	8 HIKLNOSTUWXZ	E 6	4 4 5 5 3 5 3 * 7
	I 1468	R 2679	$ 5679	9 JKLRSTVWXYZ$	E 3	3 3 3 4 5 5 6 7 *
	A 1235	J 1389	S 3457	1 ABCDEFGHIJKL	E 7	* 6 6 3 4 5 5 4 3
	B 1238	K 1456	T 3479	2 ABCDEFMNOPQR	E 7	6 * 4 5 3 5 4 6 3
I = 47	C 1245	L 1789	U 3489	3 ABGHIJMNSTUV	E 8	6 4 * 5 5 3 5 4 4
	D 1246	M 2345	V 3567	4 CDKMNOSTUWXY	E 6	3 5 5 * 7 4 3 5 4
	E 1267	N 2348	W 4568	5 ACGKMSVWXYZ$	E 5	4 3 5 7 * 6 4 3 4
DETERM =	F 1278	O 2468	X 4579	6 DEGHKOPQVWZ$	E 8	5 5 3 4 6 * 5 4 4
155654064	G 1356	P 2679	Y 4589	7 EFHILPRSTVXZ	E 6	5 4 5 3 4 5 * 3 7
	H 1367	Q 2689	Z 5679	8 BEJLNOQRUWY$	E 5	4 6 4 5 3 4 3 * 7
	I 1379	R 2789	$ 5689	9 IJLPQRTUXYZ$	E 4	3 3 4 4 4 4 7 7 *

* Represents the number 12.

Table 3 (continued)

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix
	A 1237	J 1579	S 2689	1 ABCDEFGHIJKL	E 5	* 5 4 4 4 3 6 7 3
	B 1238	K 1589	T 3456	2 ABCDEMNOPQRS	E11	5 * 5 5 5 5 3 5 3
I = 48	C 1246	L 1789	U 3469	3 ABFGMNOTUVWX	E 7	4 5 * 5 6 4 6 3 3
	D 1247	M 2345	V 3479	4 CDHIMNPTUVYZ	E 8	4 5 5 * 3 6 5 4 4
	E 1268	N 2347	W 3567	5 FGJKMOPQRTW$	E 8	4 5 6 3 * 5 4 4 5
DETERM.=	F 1357	O 2358	X 3679	6 CEHPQSTUWXY$	F 7	3 5 4 6 5 * 3 4 6
157815216	G 1358	P 2456	Y 4689	7 ADFIJLNVWXZ$	F 2	6 3 6 5 4 3 * 3 6
	H 1468	Q 2569	Z 4789	8 BEGHIKLORSYZ	E 5	7 5 3 4 4 4 3 * 6
	I 1478	R 2589	$ 5679	9 JKLQRSUVXYZ$	F 2	3 3 3 4 5 6 6 6 *
	A 1237	J 1569	S 2789	1 ABCDEFGHIJKL	E 7	* 5 6 4 6 4 3 5 3
	B 1238	K 1589	T 3456	2 ABCDEMNOPQRS	E 6	5 * 4 4 5 3 7 5 3
I = 49	C 1245	L 1689	U 3468	3 ABFGHIMNTUVW	E 9	6 4 * 6 3 4 4 5 4
	D 1247	M 2379	V 3479	4 CDFGOPTUVWXY	E 9	4 4 6 * 4 6 5 3 4
	E 1258	N 2389	W 3489	5 CEFHJKOQRTZ$	F 9	6 5 3 4 * 6 4 4 4
DETERM.=	F 1345	O 2456	X 4679	6 HIJLOPQTUXYZ	F 9	4 3 4 6 6 * 4 4 5
167546880	G 1347	P 2467	Y 4689	7 ADGMPQRSVXZ$	F 5	3 7 4 5 4 4 * 3 6
	H 1356	Q 2567	Z 5679	8 BEIKLNRSUWY$	F 6	5 5 5 3 4 4 3 * 7
	I 1368	R 2578	$ 5789	9 JKLMNSVWYZ$	F 5	3 3 4 4 4 5 6 7 *
	A 1235	J 1567	S 2458	1 ABCDEFGHIJKL	E 6	* 3 5 5 4 5 7 4 3
	B 1237	K 1678	T 2468	2 ABCMNOPQRSTU	E 6	3 * 7 5 5 4 4 5 3
I = 50	C 1257	L 1789	U 2689	3 ABDEFMNOPQVW	E 6	5 7 * 4 5 5 3 3 4
	D 1346	M 2346	V 3569	4 DEGHIMNRSTXY	F12	5 5 4 * 4 4 5 5 4
	E 1347	N 2347	W 3589	5 ACEJORSVWXYZ	F12	4 5 5 4 * 4 5 4 5
DETERM.=	F 1356	O 2358	X 4579	6 DFGJKMPTUVZ$	F12	5 4 5 4 4 * 4 5 5
187142400	G 1468	P 2369	Y 4589	7 BCEHJKLNRXZ$	F 6	7 4 3 5 5 4 * 3 5
	H 1479	Q 2389	Z 5679	8 GIKLOQSTUWY$	E 6	4 5 3 5 4 5 3 * 7
	I 1489	R 2457	$ 6789	9 HILPQUVWXYZ$	F 6	3 3 4 4 5 5 5 7 *

* Represents the number 12.

Table 2 (continued)

Case N_i^9	3-Simplices			Simplices with given vertex	Link	Edge-valence matrix								
	A - 1236	J - 1469	S - 3578	1 - ABCDEFGHIJKL	E 7	•	6	5	6	4	5	4	3	3
	B - 1238	K - 1479	T - 3579	2 - ABCDEFMNOPQR	E 7	6	•	6	5	5	4	3	4	3
I = 51	C - 1245	L - 1679	U - 3789	3 - ABGHIMNOPSTU	E 7	5	6	•	4	6	3	3	5	4
	D - 1247	M - 2345	V - 4678	4 - CDGHJKMNQVWX	E 7	6	5	4	•	3	6	5	3	4
	E - 1258	N - 2346	W - 4689	5 - CEGIMORSTYZ$	E 7	4	5	6	3	•	3	4	6	5
DETERM.=	F - 1267	O - 2359	X - 4789	6 - AFHJLNQVWYZ$	E 7	5	4	3	6	3	•	6	4	5
250838208	G - 1345	P - 2389	Y - 5678	7 - DFKLQSTUVXYZ	E 7	4	3	3	5	4	6	•	5	6
	H - 1346	Q - 2467	Z - 5679	8 - BEIPRSUVWXY$	E 7	3	4	5	3	6	4	5	•	6
	I - 1358	R - 2589	$ - 5689	9 - JKLOPRTUWXZ$	E 7	3	3	4	4	5	5	6	6	•

• Represents the number 12.

concepts are very helpful, and for their definition and significance we refer again to [4]. Although the edge-valence matrix was originally defined in [4] for 3-spheres only, the same definition holds for every 3-manifold. The 2-spheres $e_1, \ldots, e_7$ were denoted in [4] by X, Y_1, Y_2, U, Z_1, Z_2 and W, respectively.

Next we must find the $\mathcal{N}^9$'s that are not directly obtainable. This is done by a simple and natural modification of the method described in detail in [3] for finding all 3-manifolds with eight vertices. We describe here only the main idea. For further details the reader should consult [3]. Moreover, this method will yield all the neighborly 3-manifolds with nine vertices, regardless of whether or not they are directly obtainable.

Let M be a neighborly 3-manifold with the nine vertices $1, 2, \ldots, 9$. Double counting of the 2-simplices in M and (*) show that M contains precisely 27 3-simplices. The 2-sphere S = link $(9, M)$ has eight vertices and is therefore one of $e_1, \ldots, e_{14}$ in Fig. 1. Since S contains 12 triangles, star $(9, M)$ contains 12 3-simplices and therefore R = antistar $(9, M)$ contains 15 3-simplices. Notice that star $(9, M)$ depends on S only, namely, it is $9 \vee S$. We denote it by S'.

For each of the 14 possibilities for S, we label the vertices of S as $1, 2, \ldots, 8$, and we find all possible 3-complexes R with the same eight vertices such that $R \cup S'$ is a neighborly 3-manifold. Following [3, Section 3], R is easily seen to satisfy the following conditions:

(1) $f_3(R) = 15$.

(2) Each 2-simplex $\Delta \in R$ belongs to precisely two 3-simplices in R if $\Delta \notin S$ and to precisely one 3-simplex in R if $\Delta \in S$.

(3) For each vertex $x \in S$, if x is of valence j in S, then x belongs to precisely $12-j$ 3-simplices in R.

(4) $S \subset R$.

(5) For every two vertices $x, y \in R$, there is in R a 3-simplex that contains both x and y.

After obtaining all such complexes R (again, consult [3] for details), a check was made to verify that each $S' \cup R$ was indeed a 3-manifold. It is interesting to note that for each S and for each R that corresponds to that S, $R \cup S'$ was indeed a manifold (see [3, Remark 4.2]). For each of the manifolds thus obtained, the edge-valence matrix and its determinant were calculated, as well as the links of the vertices. All manifolds having the same determinant were checked, and were found to be isomorphic to each other. Altogether, 51 isomorphism classes were found and a representative for each class was selected.

The 50 directly obtainable N^9's previously obtained are easily identified in the new list by the determinant of the edge-valence matrix. The remaining 3-manifold N_{51}^9 (see Table 2), has the highest determinant of the 51 cases. A priori, all that we know about N_{51}^9 is that it is not a directly obtainable sphere, and therefore it may be either a sphere that is not directly obtainable, or a manifold that is not a sphere. However, it is easy to check that N_{51}^9 is not orientable, and hence it is not a sphere. The 3-simplices M, N, O, Q, S, T, V and Y of N_{51}^9 form a solid Klein bottle K. The boundary complex of K is a 2-dimensional Klein bottle with 8 vertices, 24 edges and 16 triangles, and is therefore a minimal triangulation of a topological Klein bottle (see [9]).

The proof of Theorem 1.1, and therefore also of Theorem 1.2, is thus complete.

4. Remarks

(1) Each 2-sphere e_i in Fig. 1 is accompanied by the 8-tuple of the valences of the vertices in e_i (the 8-tuple that corresponds to e_i ($1 \le i \le 7$) has been denoted by C_i in [4]). For each N^9 and each vertex $v \in N^9$, the 8-tuple that corresponds to link (v, N^9) is directly readable from the edge-valence matrix of N^9 in the manner explained in [4]. Notice that e_7 and e_{14} share the same 8-tuple, this accounting for the fact that the 3-sphere N_{45}^9 appeared in [4] (where it is denoted by N) although it is not polytopal.

(2) Table 3 summarizes the types of links corresponding to the vertices of N_i^9 ($24 \le i \le 51$), and is thus a continuation of [4, Table 3] where a similar summary for the polytopal spheres $N_1^9, \ldots, N_{23}^9$ was given. From this summary, we see that no N^9 has a vertex v such that link (v, N^9) is isomorphic to e_{13}, i.e., to the boundary complex of a 3-dimensional bipyramid with 8 vertices. From [10] and [12], we see that neighborly 3-manifolds with seven or eight vertices possess a similar property. This leads us to venture the following conjecture:

Table 3
The links[a] of the vertices of the manifolds N_i^9, $24 \leq i \leq 51$ (summary).

N_i^9 i	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}
24	2				6						1			
25		2	4				1		2					
26			1	1	4	1		1						1
27		2	3		3				1					
28			4		2		1		2					
29			2		2	2			2			1		
30		2	3		1		2		1					
31			1	1	1	4		1		1				
32					4		1	2		2				
33		2	2		2	2					1			
34		1	1		1	2	1	1	1					1
35		2	1		3	2			1					
36			1		5	2			1					
37		2	2				2			2	1			
38		1	2		1	1	2		1					1
39		2			1	1	1	1	3					
40			2		2	2	1				1			1
41		1	1		3		2			1				1
42		2	1		1		2		3					
43			1		4	1	1		1	1				
44					6		1		2					
45					4	2	1							2
46			2			2	1			2	2			

Table 3 (continued)

N_i^9	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}
47				1	2	2	2	2						
48		2			2		2	2			1			
49					2	2	1		4					
50						6						3		
51							9							

Conjecture 4.1. *For every neighborly 3-manifold N with more than 6 vertices and for every vertex $v \in N$, link (v, N) is not isomorphic to the boundary complex of a 3-dimensional bipyramid.*

From [4, Theorem 1] it follows that for each polytopal neighborly 3-sphere N and for each vertex $v \in N$, the 2-sphere link (v, N) is stacked. This does not mean that the same property cannot be shared by non-polytopal neighborly 3-spheres. However, another conclusion from Table 3 is that every non-polytopal sphere N^9 has at least one vertex v such that link (v, N^9) is not stacked. Also, a close examination of the 3-spheres N_i^9 $(1 \leq i \leq 50)$ shows that for each $1 \leq i \leq 50$ and for each vertex $v \in N_i^9$ such that link (v, N_i^9) is stacked, the sphere N_i^9 is directly obtainable at v. The last two properties imply that each sphere N_i^9 is directly obtainable. (This result can also be directly derived from Section 3.) One cannot help but wonder whether or not any of these properties is shared by all neighborly 3-spheres, regardless of the number of vertices. In particular, if it is true that every 3-sphere (not necessarily neighborly) is directly obtainable, then, in view of the famous Poincare conjecture, it would be very helpful in checking whether or not a given (combinatorial) homotopy 3-sphere is a sphere (see [10, Conjecture 5]).

(3) We now turn to a closer examination of the most interesting case N_{51}^9. The fact that all nine link (v, N_{51}^9) (v is a vertex in N_{51}^9) are of the same type already indicates that N_{51}^9 possesses a high degree of symmetry. Indeed, each of the permutations

$$\lambda = (2, 4)(3, 6)(5, 7)(8, 9), \qquad \psi = (1, 3)(4, 5)(6, 8)(7, 9)$$

of the nine vertices of N_{51}^9 induces a combinatorial automorphism of N_{51}^9 which is of period 2, and using products of these two automorphisms it is easily checked that for every two vertices $x, y \in N_{51}^9$, there is a combinatorial automorphism of N_{51}^9 that maps x to y.

For each vertex $v \in N_{51}^9$, $S = \text{link}(v, N_{51}^9)$ is isomorphic to the stacked 2-sphere e_7. Since by property (1) in Section 2, the 3-element C associated with S in the manner mentioned there does exist and is unique, the only reason that N_{51}^9 is not directly obtainable at v is that $C \cap \text{antistar}(v, N_{51}^9) \neq S$. Indeed, for $v = 1$, for example, the 3-simplices of the corresponding C are 2467, 2346, 2345, 4679 and 2358, and the first three are in antistar$(1, N_{51}^9)$. Notice that the 3-simplices of C that are not in antistar$(1, N_{51}^9)$ contain the vertices that are of valence 3 in link$(1, N_{51}^9)$.

(4) It follows from the present work that the determinant of the edge-valence matrix does discriminate between the different N^9's. This leads us to generalize [4, Conjecture 1].

Conjecture 4.2. *Non-isomorphic neighborly 3-manifolds with the same number of vertices have distinct determinants.*

It is possible that further investigation of the edge-valence matrix will reveal some geometric or topological significance of its determinant. One can show that the determinant of each of the N^9's is divisible by 144 as follows. Each of the main diagonal entries in the edge-valence matrix is 12. The sum of all the other entries in each row and in each column is 36 (see [4] for more details). Adding the sum of all the other rows to the last row makes each entry in the last row 48. Adding the sum of all the other columns to the last column makes each entry in the last column 48, except the last entry which becomes $9 \cdot 48$. Hence the determinant is divisible by $48 \cdot (9, 48) = 144$ (where (x, y) denotes the greatest common divisor of x and y). A similar computation shows that the determinant of the edge-valence matrix of every neighborly 3-manifold with n vertices is divisible by $(8n - 24) \cdot (n, 8n - 24) = (8n - 24) \cdot (n, 24)$.

(5) In view of Theorem 1.2, it is interesting to speculate about the minimal n for which there exists an orientable 3-manifold with n vertices that is not a sphere. Such an n is of course greater than eight, but the present work does not exclude the possibility that it is nine.

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